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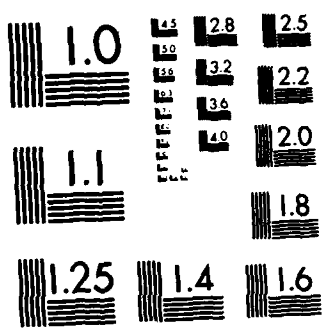

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Initially the problem was to study the dominates relation on a collection of semigroup operations called triangular norms. This led to an equivalent problem --- studying subadditivity of certain semigroup operations defined on the non-negative reals. The setting was later generalized to include both problems and to bring essentials of the problem into sharper focus. In the generalization, a partially ordered set S was endowed with the collection, Op(s), of all semigroup operations which had the same identity, e, and were non-decreasing in place. The dominates relation was defined on Op(S). (CONTINUED)

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ITEM #18, CONTINUED: Oberwolfach, West Germany, in August 1982.

ITEM #20, CONTINUED: The collection, Map(S), of order-preserving bijections from S to S which map e to itself was used to partition Op(S) into equivalence classes — two objects being placed in the same class if they were isomorphic via some member of Map(S). Dominates restricted to any equivalence class in Op(S) was shown to to exhibit a certain homogeniety relative to composition of elements in Map(S). Transitivity of dominates on any equivalence class was shown to be equivalent to an appropriate subset of Map(S) being algebraically closed under composition. The equivalence classes determined by continuous triangular norms were characterized in terms of ordinal sums of semigroups.

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DOMINATES ON EQUIVALENCE CLASSES
OF SEMIGROUP OPERATIONS

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DOMINATES ON EQUIVALENCE CLASSES OF SEMIGROUP OPERATIONS

1. Preliminaries. Throughout this report $(S,\geq)$ always denotes a partially ordered set and e denotes a fixed element of S. Particular realizations of importance are (1) $S = I = [0,1]$ endowed with the usual ordering and $e = 1$, (2) $S = \mathbb{R}^+ = [0,\infty]$ with the usual ordering and $e = 0$, (3) $S = \Delta^+$, the collection of all nondecreasing, left-continuous functions F from $\mathbb{R}^+$ into I with $F(0) = 0$ and $F(\infty) = 1$ where for any F, G in Δ^+, $F \geq G$ if and only if $F(x) \geq G(x)$ for all x in $\mathbb{R}^+$, and e is taken to be the element ε_0 in Δ^+ having a unit jump discontinuity at zero.

By Op(S) is meant the collection of all associative binary operations on S having e as an identity. The elements of $Op(\Delta^+)$ which are commutative and nondecreasing in each place are called triangle functions [3]* while those in Op(I) which are both commutative and nondecreasing in each place are called triangular norms (briefly, t-norms) [3]. A t-norm is strict if it is continuous on I^2 and strictly increasing in each place on $(0,1]^2$. A t-norm, T, is Archimedean [3] if and only if, for each x with $0 < x < 1$, $T(x,x) < x$.

A family $\{T_p\}_{p=-\infty}^{\infty}$ of t-norms of special interest in this report is defined as follows: For each real number $p \neq 0$, let $T_p\colon I^2 \to I$ be given by

$$T_p(a,b) = [\mathrm{Max}(a^p + b^p - 1,0)]^{1/p} \tag{1.1}$$

where zero raised to a real power is understood to be zero. Define

*Schweizer, B. and Sklar, A., Probabilistic Metric Spaces, Elsevier North Holland, New York (to appear Fall 1982)

$$T_0(a,b) = a \cdot b \tag{1.2}$$

$$T_{-\infty}(a,b) = \mathrm{Min}(a,b), \tag{1.3}$$

and

$$T_\infty(a,b) = \begin{cases} a, & \text{if } b = 1 \\ b, & \text{if } a = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1.4}$$

The t-norm T_p is continuous if and only if $-\infty \le p < \infty$, strict if and only if $-\infty < p \le 0$, and Archimedean if and only if $-\infty < p \le \infty$. Moreover $T_p \to T_0$ as $p \to 0$, $T_p \to T_\infty$ as $p \to \infty$ and $T_p \to T_{-\infty}$ as $p \to -\infty$. The t-norms $T_{-\infty}$, T_0, T_1 and T_∞ are important t-norms whose standard names in the literature [3] are M, Π, W and Z, respectfully.

It is well-known [3] that a t-norm T is strict if and only if there is a continuous, strictly decreasing function g from I onto $\mathbb{R}^+$ such that for each a,b in I,

$$T(a,b) = g^{-1}(g(a) + g(b)). \tag{1.5}$$

The function g is called an <u>additive generator</u> for T. Moreover, if g satisfies the above conditions, then T given by (1.5) is a strict t-norm.

The ordering on S induces two relations of interest on Op(S) -- one is the <u>stronger than or equal to</u> relation, the other is the <u>dominates</u> relation. For any H,G in Op(S), H is <u>stronger than or equal to</u> G (written $H \ge G$) if and only if, for all a and b in S, $H(a,b) \ge G(a,b)$.

Also, for any H,G in Op(S), H <u>dominates</u> G (written $H \gg G$) if and only if, for all a, b, c, d in S,

$$H(G(a,b),G(c,d)) \geq G(H(a,c),H(b,d)) \qquad (1.6)$$

The definition of dominates in this setting is formulated in [3] where its importance is discussed and some of the problems in the next section are posed.

When forming the product of two metric spaces (M_1,d_1) and (M_2,d_2), the dominates relation on $Op(\mathbb{R}^+)$ is interesting. For an arbitrary H in $Op(\mathbb{R}^+)$, the function $d: (M_1 \times M_2)^2 \to \mathbb{R}^+$ defined by

$$d((p_1,p_2),(q_1,q_2)) = H(d_1(p_1,q_1),d_2(p_2,q_2))$$

might not be a metric because the triangle inequality may fail. If, however, ordinary addition dominates H then the triangle inequality is satisfied and d is a metric.

Similarly, when forming the product of two probabilistic metric spaces [6]* the dominates relation on Op(I) and the dominates relation on $Op(\Delta^+)$ are interesting. The principle investigator became interested in dominates on Op(I) when studying products of fuzzy subgroups [4]**.

* Tardiff, R. M., Topologies for probabilistic metric spaces, Ph.D. dissertation, University of Massachuesetts, 1975 (Unpublished)

** Sherwood, H., Products of fuzzy subgroups. <u>Fuzzy Sets and Systems</u> (To appear)

2. Research Objectives. All of the objectives of this project dealt with t-norms. One of the main objectives was to seek a condition on t-norms T and R which would imply that T dominates R. To find an easier-to-apply condition equivalent to (1.6) was a more ambitious objective. Other objectives were (1) to determine whether $T \gg R$, (2) to determine whether dominates is a transitive relation, and (3) to study the connections between dominates on t-norms and dominates on triangle functions. In order to gain insight into the above-mentioned problems, the very first objective was to characterize dominates on the family $\{T_p\}_{p=-\infty}^{\infty}$. Next, the strict t-norms were to be considered.

3. Status of Research. The initial objective of this project, characterizing dominates on the family $\{T_p\}_{p=-\infty}^{\infty}$ has been achieved. A paper [5]* containing this characterization has been written which is briefly summarized in the following theorem and corollary.

Theorem 1. The t-norm T_p dominates T_q if and only if $p \le q$.

Corollary 1.1. The dominates relation is transitive on $\{T_p\}_{p=-\infty}^{\infty}$.

The work leading to these results gave hope that the same methods could be used to attack the "dominates problem" in the family of strict t-norms. As this was undertaken, equation (1.5) led to considering the equivalent problem in $\mathrm{Op}(\mathbb{R}^+)$ instead of in $\mathrm{Op}(I)$. After considerable work in that

* Sherwood, H., Characterizing dominates on a family of triangular norms, Submitted to Aequationes Mathematicae.

context, additional insight suggested working in the more abstract setting of Op(S). To communicate the results obtained, some additional conventions are needed.

Let Map(S) denote the collection of all order-preserving bijections from S onto S which map e to itself. Observe that Map(S) is a group under composition. For each H in Op(S) and each α in Map(S), the function $H_\alpha : S^2 \to S$, defined by

$$H_\alpha(a,b) = \alpha^{-1}H(\alpha(a),\alpha(b)),$$

is again in Op(S). Moreover, if H is commutative so is H_α, and if H is nondecreasing in each place so is H_α. For any H, G in Op(S), H is equivalent to G (written $H \sim G$) if and only if there is some α in Map(S) such that $G = H$. Observe that $\sim$ is an equivalence relation on Op(S) and [H] denotes the equivalence class determined by H.

In another context R. Moynihan [2]* proves that $[M] = \{M\}$, $[\Pi]$ is the collection of strict t-norms, and [W] is the collection of non-strict, Archimedean t-norms. It is easy to prove that [M] and [Z] are the only singleton equivalence classes determined by t-norms. Later in this report a characterization of all equivalence classes determined by continuous t-norms is given.

Earlier mention is made of the equivalence of the "dominates problem" for strict t-norms and another one in the setting of $Op(\mathbb{R}^+)$. This equivalence is now made precise.

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*Moynihan, R., On τ_T semigroups of probability distribution functions II. Aequations Mathematicae, 17 (1978) 19-40.

Theorem 2. Let $(S,\geq)$ and $(S',\geq')$ be partially ordered sets with distinguished elements e and e', respectively. Let H and H' denote fixed elements of S and S' and let $\geq$ and $\geq'$ denote the stronger than or equal to relations induced on $Op(S)$ and $Op(S')$, respectively. Also let $>>$ and $>>'$ denote the corresponding dominates relations on $Op(S)$ and $Op(S')$. Suppose there exists a bijection $f: S \to S'$ such that $f(e) = e'$ and for all a,b in S, (1) $a \geq b$ if and only if $f(a) \geq' f(b)$ and (2) $H(a,b) = f^{-1} H'(f(a),f(b))$. Then the map Φ: $Map(S) \to Map(S')$ defined by $\Phi(\alpha) = f \circ \alpha \circ f^{-1}$ is an isomorphism from $(Map(S),\circ)$ onto $(Map(S'),\circ)$ and the map $F: [H] \to [H']$ defined by $F(H_\alpha) = H'_{f\circ\alpha\circ f^{-1}}$ is a bijection such that, for each H_α, H_β in $[H]$, (a) $H_\alpha \geq H_\beta$ if and only if $F(H_\alpha) \geq' F(H_\beta)$ and (b) $H_\alpha >> H_\beta$ if and only if $F(H_\alpha) >>' F(H_\beta)$.

Theorem 3. Let $(S,\geq)$, $(S',\geq')$, e, e', H, H' and the induced relations $\geq$, $\geq'$, $>>$ and $>>'$ be as in Theorem 2. Suppose there exists a bijection $f: S \to S'$ such that $f(e) = e'$ and, for all a,b in S, (1) $a \geq b$ if and only if $f(b) \geq' f(a)$ and (2) $H(a,b) = f^{-1}H'(f(a),f(b))$. Then the mapping Φ: $Map(S) \to Map(S')$ defined by $\Phi(\alpha) = f\circ\alpha\circ f^{-1}$ is again an isomorphism from $(Map(S),\circ)$ onto $(Map(S'),\circ)$ and the map F: $[H] \to [H']$ defined by $F(H_\alpha) = H'_{f\circ\alpha\circ f^{-1}}$ is a bijection and such that, for each H_α, H_β in $[H]$, (a) $H_\alpha \geq H_\beta$ if and only if $F(H_\beta) \geq' F(H_\alpha)$ and (b) $H_\alpha >> H_\beta$ if and only if $F(H_\beta) >>' F(H_\alpha)$.

Consider the following special case of Theorem 3. Let f: $I \to \mathbb{R}^+$ be given by $f(x) = -\ln x$. Then, for any x,y in I,

$$\Pi(x,y) = x \cdot y = f^{-1}(f(x) + f(y)).$$

Observe that Π is in Op(I) and A: $(\mathbb{R}^+)^2 \to \mathbb{R}^+$ defined by $A(x,y) = x + y$ is in $Op(\mathbb{R}^+)$. Also, the conditions of Theorem 3 are satisfied by f, Π and A. Therefore, $\Pi_\alpha >> \Pi_\beta$ if and only if $A_{f\circ\beta\circ f^{-1}} >> A_{f\circ\alpha\circ f^{-1}}$ so that solving the "dominates problem" on $[\Pi]$ is in fact equivalent to solving it on $[A]$.

The next few results reveal an intimate connection between the group structure in (Map(S), $\circ$) and the dominates relation.

<u>Theorem 4</u>. For any H in Op(S) and any α,β,γ in Map(S), $H_\alpha >> H_\beta$ if and only if $H_{\alpha\circ\gamma} >> H_{\beta\circ\gamma}$.

<u>Corollary 4.1.</u> For any H in Op(S) and any α,β,γ in Map(S), the following three statements are equivalent: (1) $H_\alpha >> H_\beta$, (2) $H_{\alpha\circ\beta^{-1}\circ\gamma} >> H_\gamma$, and (3) $h_\gamma >> H_{\beta\circ\alpha^{-1}\circ\gamma}$.

Notice the homogeneity in the structure ($[H]$, >>). No matter where you stand in the structure, whether at H_β or at H_γ, you see essentially the same view. Also, finding all the H_α's that dominate a given H_β is accomplished as soon as all those that dominate (or are dominated by) any other fixed H_γ are found.

For some time it has been known [6] that the dominates relation is intimately connected to the notion of subadditivity and superadditivity [1]*

*Marshall, A. W. and Olkin, I., <u>Inequalities: Theory of Majorization and Its Applications</u>, Academic Press, New York, 1979.

of functions. A special case of Corollary 4.1 brings out this connection.

Corollary 4.2. Let A in $Op(\mathbb{R}^+)$ denote ordinary addition. For any α, β in $Map(\mathbb{R}^+)$, the following three statements are equivalent: (1) $A_\alpha >> A_\beta$, (a) $A_{\alpha \circ \beta^{-1}} >> A$, i.e., $A_{\alpha \circ \beta^{-1}}$ is superadditive, and (3) $A >> A_{\beta \circ \alpha^{-1}}$ is subadditive.

The structure ([H], $\geq$) possesses the *same* sort of homogeniety as ([H], >>). In other words, $H_\alpha \geq H_\beta$ if and only if $H_{\alpha \circ \gamma} \geq H_{\beta \circ \gamma}$. The homogeniety of these two structures coupled with the following example leads to an interesting result.

Example. Let A in $Op(\mathbb{R}^+)$ denote ordinary addition and let γ in $Map(\mathbb{R}^+)$ be given by

$$\gamma(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1, \\ (x+1)/2, & \text{if } x > 1. \end{cases}$$

Then A is given by a multipart rule whose value at any (x,y) in the various regions of $(\mathbb{R}^+)^2$ are shown in Figure 1. From Figure 1 it is clear that $A_\gamma \geq A$. But $A_\gamma(2,2) = 5$ while $A_\gamma(1,1) = 3$.

$$\begin{aligned} A_\gamma(A(1,1),A(1,1)) &= A_\gamma(1+1,\ 1+1) = 5 \\ &< 3+3 = A_\gamma(1,1) + A_\gamma(1,1). \end{aligned}$$

Thus A_γ does not dominate A.

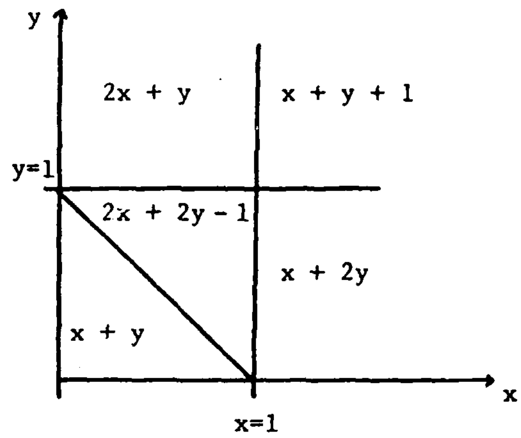

Figure 1

Theorem 5. For every A_α there is an A_β such that $A_\beta \geq A_\alpha$ but A_β does not dominate A_α.

The transitivity of dominates is also intimately connected with the group structure in (Map(S),$\circ$) as indicated by the following theorem and corollary.

Theorem 6. Dominates is transitive on [H] if and only if whenever $H_\alpha >> H$ and $H_\beta >> H$ then $H_{\alpha\circ\beta} >> H$.

If [H] inherits the group operation from Map(S), i.e., $H_\alpha * H_\beta = H_{\alpha\circ\beta}$, then Theorem 6 yields the following corollary.

Corollary 6.1. Dominates is transitive on [H] if and only if $\{H_\alpha : H_\alpha >> H\}$ is closed under $*$.

In order to characterize all the equivalence classes determined by

continuous t-norms, the following definition taken from [3] is needed.

Definition 1. Let $\{(S_i,T_i)\}_{i\in T}$ be a family of binary systems indexed by the elements of a (possibly uncountable) set T of real numbers. For all i, j, k in T, let these binary systems satisfy the following compatibility conditions:

(1) If $i < j < k$ and $S_i \cap S_k$ is non-empty, then $S_j = S_i \cap S_k$,

(2) If $i < j$ and x is in $S_i \cap S_j$, then x is the unique element in $S_i \cap S_j$, and is the identity of T_i and the null element of T_j.

Finally, let S be the union of all sets S_i, and let T be defined on S^2 as follows: For x in S_i and y in S_j,

$$T(x,y) = \begin{cases} x, & \text{if } i < j, \\ T_i(x,y), & \text{if } i = j, \\ y, & \text{if } i > j. \end{cases} \tag{3.1}$$

(Note that the compatibility conditions guarantee that (3.1) is consistent, whence T is a well-defined binary operation on S.) Then the binary system (S,T) is the ordinal sum of the family of binary systems $\{(S_i,T_i)\}_{i\in T}$.

In [3] Schweizer and Sklar state a theorem, which in the continuous case they credit to Mostert and Shields, stating that under suitable conditions a binary system may be expressed as an ordinal sum. In general this representation is not unique. However, the proof of Theorem 5.3.8.

in [3] reveals that under the conditions of that theorem it is always possible to choose a standard way of doing this. These observations motivate the following definition and theorem.

Definition 2. The family $([t_i,e_i],T_i)_{i\in T}$ is a canonical ordinal sum family for the continuous t-norm T if and only ..

(1) For every i in T, $i = (t_i + e_i)/2$,

(2) Each T_i is either Archimedean on $[t_i,e_i]$ or $T_i = M|[t_i,e_i]^2$

(3) Each $[t_i,e_i]$ for which $T_i = M|[t_i,e_i]^2$ is maximal in the sense that if it were enlarged then there would be a point x in the enlargement where $T(x,x) < x$, and

(4) $([0,1],T)$ is an ordinal sum of the family $([t_i,e_i],T_i)_{i\in T}$.

Theorem 7. Every continuous t-norm admits exactly one canonical ordinal sum family.

Figure 2 illustrates the situation described in the preceding definition and theorem. On the squares labelled "A", T is Archimedean while everywhere else in I^2, T equals Min.

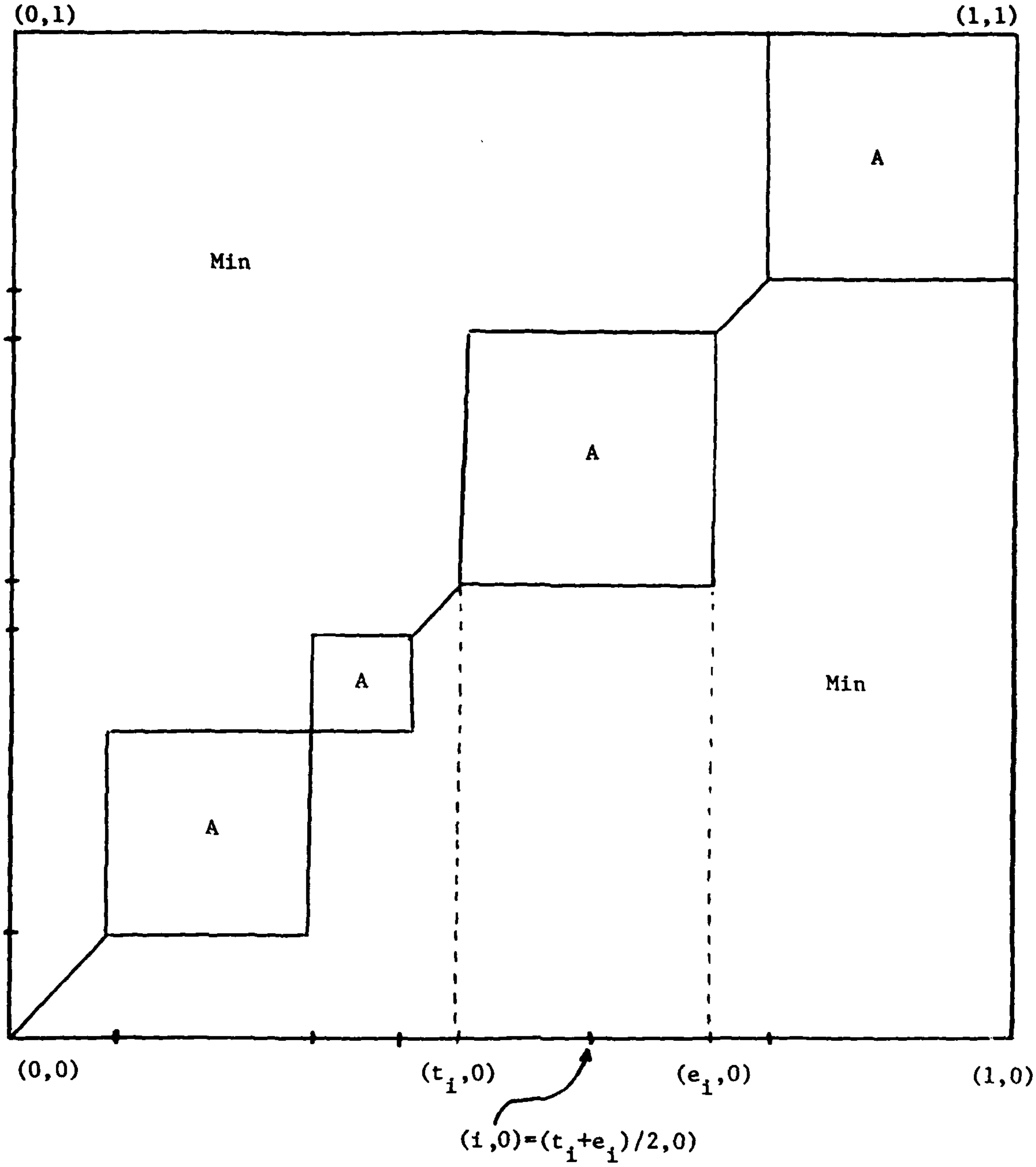

Figure 2

Definition 3. Let T be a continuous t-norm with canonical ordinal sum family $([t_i,e_i],T_i)_{i\in T}$. The map δ_T: $T \to \{0,1,2,3\}$ is defined as follows:

$$\delta_T(i) = \begin{cases} 0, & \text{if } t_i = e_i \\ 1, & \text{if } T_i = M|[t_i e_i]^2 \text{ and } t_i \neq e_i, \\ 2, & \text{if } T_i \text{ is strict and } t_i \neq e_i, \\ 3, & \text{otherwise.} \end{cases}$$

Theorem 8. The continuous t-norms T and R are equivalent (i.e., $T \sim R$) if and only if there exists an order-preserving bijection f from the domain of δ_T onto the domain of δ_R such that $\delta_T = \delta_R \circ f$.

Corollary 8.1 Every continuous t-norm is equivalent to an ordinal sum of semigroups (S_i,T_i) where each S_i is a closed interval $[t_i,e_i]$ (perhaps consisting of a single point in which case $T_i(t_i,t_i) = t_i$) and each T_i (for which $t_i \neq e_i$) is given by

$$T_i(x,y) = (e_i-t_i)R_i((x_i-t_i)/(e_i-t_i),(y_i-t_i)/(e_i-t)) + t_i,$$

for each x,y in $[t_i,e_i]$ where R_i is W, Π or M.

In light of Theorem 4 and the corollaries to it, finding conditions which guarantee that one member of [A] dominates another reduces to finding conditions which guarantee a member to be subadditive. Condition (1.6) as applied to members of [A] is a four-variable condition. The next lemma replaces condition (1.6) with infinitely many two-variable extremal problems.

<u>Lemma</u> For each (x_o,y_o) in $(\mathbb{R}^+)$ define $G_{(x_o,y_o)}$: $[0,x_o] \times [0,y_o] \to \mathbb{R}^+$ via

$$G_{(x_o,y_o)}(x,y) = A_\alpha(x,y) + A_\alpha(x_o-x,y_o-y).$$

Then A is subadditive if and only if each $G_{(x_o,y_o)}$ assumes its minimum value at $(x,y) = (x_o,y_o)$.

This lemma is used to prove the following result.

<u>Theorem 9</u>. Suppose α is an everywhere differentiable, strictly increasing map from $\mathbb{R}^+$ onto $\mathbb{R}^+$ such that for every (x_o,y_o), the only solution of

$$\frac{\alpha'(x)}{\alpha'(x_o-x)} = \frac{\alpha'(y)}{\alpha'(y_o-y)} = \frac{\alpha'(\alpha^{-1}(\alpha(x)+\alpha(y)))}{\alpha'(\alpha^{-1}(\alpha(x_o-x)+\alpha(y_o-y)))} \tag{3.2}$$

is $(x,y) = (x_o/2,y_o/2)$. Then A_α is subadditive if and only if α is convex and the map $x \mapsto \alpha(2\alpha^{-1}(x))$ is superadditive.

The condition that (3.2) have only one solution is not needed to prove that the subadditivity of A_α guarantees the convexity of α and the subadditivity of $x \mapsto \alpha(2\alpha^{-1}(x))$. Furthermore, examples exist where A_α is subadditive and (3.2) has more than one solution. However, for each known example of a subadditive A_α, it happens that α is convex and $x \mapsto \alpha(2\alpha^{-1}(x))$ is superadditive. A proof that these two conditions alone are sufficient for the subadditivity of A_α is yet to be found.

4. Conclusions Because the objectives for this project are stated in the context of t-norms and because many of the above results are stated either in a different or more general setting, the progress toward the stated objectives is now summarized in terms of t-norms.

First of all Theorem 9, in light of the discussion immediately following Theorem 3, gives in many situations an easier to apply condition equivalent to (1.6) for strict t-norms. The condition is not as easy to apply as desired, but it warrants further study especially in the direction suggested by the discussion following Theorem 9. Thus, some progress has been made toward the main objective.

Determining whether $T \geq R$ implies $T \gg R$ was another objective. Unknown to the principle investigator this first objective had already been resolved by R. Tardiff [6] with a counterexample. However, again in light of the discussion immediately following Theorem 3, Theorem 5 addresses this objective much more forcefully than a single counterexample; it says that given any strict t-norm R there is a strict t-norm T such that $T \geq R$ but T does not dominate R.

Whether dominates is transitive is still an open question in general. On the family $\{T_p\}_{p=-\infty}^{\infty}$ this has been resolved in the affirmative. Moreover, Corollary 6.1 reduces the problem for strict t-norms to a closure-type problem, i.e., dominates is transitive on the collection of strict t-norms if and only if $\{\Pi_\alpha : \Pi_\alpha \gg \Pi\}$ is closed under $*$.

Finally, the very first objective, characterizing dominates on the family $\{T_p\}_{p=-\infty}^{\infty}$, has been completely realized. As mentioned earlier, these results are the subject of a paper [5].

References

[1]. Marshall, A.W. and Olkin, I., Inequalities: Theory of Majorization and Its Applications, Academic Press, New York, 1979.

[2]. Moynihan, R., On τ_T semigroups of probability distribution functions II. Aequationes Math., 17 (1978) 19-40

[3]. Schweizer, B. and Sklar, A., Probabilistic Metric Spaces, Elsevier North Holland, New York (to appear fall 1982)

[4]. Sherwood, H., Products of fuzzy subgroups. Fuzzy Sets and Systems (to appear).

[5]. Sherwood, H., Characterizing dominates on a family of triangular norms, Submitted to Acquationes Mathematics.

[6]. Tardiff, R. M., *Topologies for probabilistic metric spaces*, Ph.D. dissertation, University of Massachusetts, 1975 (Unpublished)

List of Written Publications

1. "Characterizing Dominates on a Family of Triangular Norms". The manuscript is being submitted for publication to Aequationes Mathematicae

2. Other publications may result from these efforts but they may be joint works with R. Tardiff. There is some overlap in our work and it may not be possible to sort out who obtained which results first.

Interactions

The principle investigator was invited to present a paper on some of the results associated with this research at the 19th and at the 20th International Symposium on Functional Equations. The paper resented at the 19th Symposium in Brittany, France in May 1981 was entitled "'Dominates', a relation on triangular norms." Travel to that symposium was paid for by the University of Central Florida. The paper presented at the 20th Symposium in Oberwolfach, West Germany in August 1982 was entitled "Dominates on equivalence classes of operations." Travel to the 20th Symposium was paid for out of these grant funds.

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